11. Taylor Polynomial, Extreme Values Part 1

In this lecture, we will discuss

- Second Degree Taylor Polynomial
- Extreme values Part 1

Remark.

- For the homework in this section, we mainly focus on Extreme Values Part 1. The questions are mostly a review of problems on extreme values for single variable functions.
- We will discuss extreme values for multivariable ~~variable~~ functions in the next lecure, which will be in Lecture 12.

Second Degree Taylor Polynomial

Review of Taylor Polynomials for a Function of One Variable

Definition. Taylor polynomials for a function of one variable, $y = f(x)$

If f has n derivatives at $x = a$ then the polynomial,

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

is called the n^{th} -degree Taylor Polynomial for f at a.

Remark.

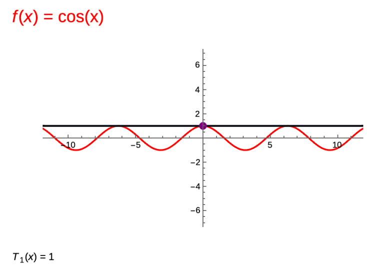
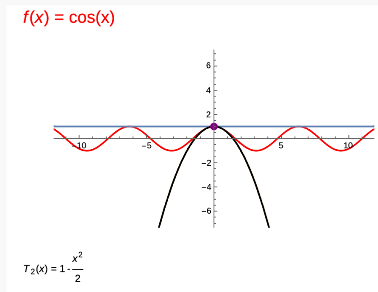
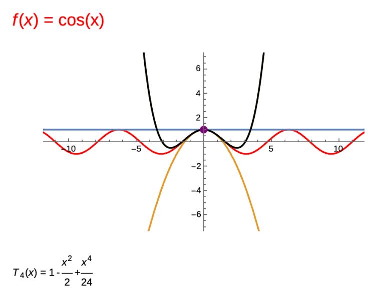
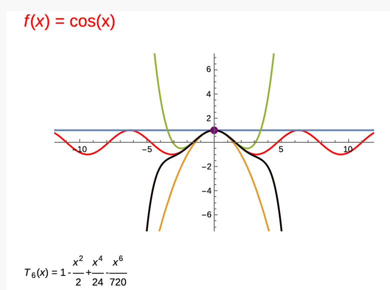
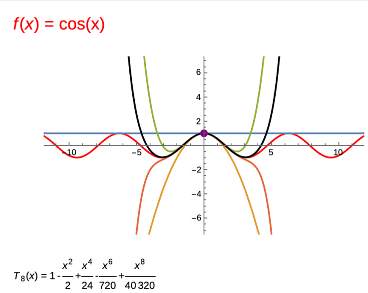
1. The 1^{st} -degree Taylor Polynomial is also called the linear approximation of $f(x)$ for x near a.

$$f(x) \approx f(a) + f'(a)(x-a)$$

2. The 2^{nd} -degree Taylor polynomial of f at a is also called the quadratic approximation

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Here is a table shows the graph of $f(x) = \cos x$ and its Taylor polynomials at $a = 0$ for different degrees. We can see that as n Increases, we get better estimations.

n	**Graphs for $f(x) = \cos x$ and $T_n(x)$**
1	
2	
4	
6	
8	

First-degree Taylor polynomial of a function of two variables, $f(x,y)$ (Review)

For a function of two variables $f(x,y)$ whose first partials exist at the point (a,b), the 1^{st} -degree Taylor polynomial of f for (x,y) near the point (a,b) is:

$$f(x,y) \approx L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$L(x,y)$ is called the linear (or tangent plane) approximation of f for (x,y) near the point (a,b), which we discussed in Lecture 5.

Definition: Second-degree Taylor Polynomial of a function of two variables, $f(x,y)$

For a function of two variables $f(x,y)$ whose first and second partials exist at the point (a,b), the 2^{nd} degree Taylor polynomial of f for (x,y) near the point (a,b) is:

$$f(x,y) \approx Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

Note we can simplify this formula as is we already know $L(x,y)$:

$$f(x,y) \approx Q(x,y) = L(x,y) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{f_{yy}(a,b)}{2}(y-b)^2$$

Example 1 Determine the 1^{st} - and 2^{nd} -degree Taylor polynomial approximations, $L(x,y)$ and $Q(x,y)$, for the function $f(x,y) = xe^y + 1$ for (x,y) near the point $(1,0)$

ANS: By the definition, we need to compute.

$$f_x(x,y) = e^y \quad \text{and} \quad f_y(x,y) = xe^y$$

and

$$f(1,0) = 1e^0 + 1 = 2$$

$$f_x(1,0) = e^0 = 1$$

$$f_y(1,0) = 1e^0 = 1$$

Thus $L(x,y) = f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0)$

$$= 1 + x + y$$

Then we compute:

$$f_{xx}(x,y)=0, \quad f_{xy}(x,y)=e^{y}, \quad f_{yy}(x,y)=xe^{y}$$

and

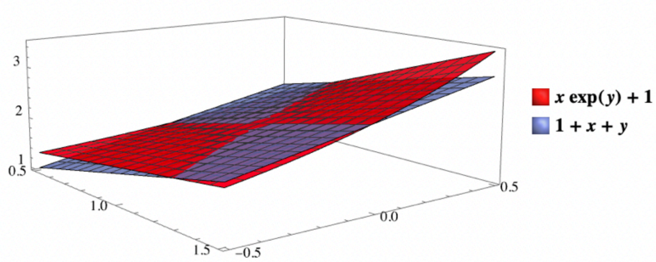
$$f_{xx}(1,0)=0, \quad f_{xy}(1,0)=1, \quad f_{yy}(1,0)=1.$$

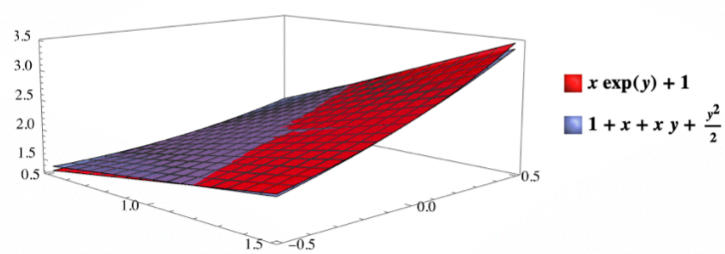
Then

$$Q(x,y) = L(x,y) + \frac{f_{xx}(1,0)}{2}(x-1)^2 + f_{xy}(1,0)(x-1)(y-0) + \frac{f_{yy}(1,0)}{2}(y-0)^2$$

$$= 1+x+y+xy-y+\frac{y^2}{2}$$

$$= 1+x+xy+\frac{y^2}{2}.$$





For the rest of this lecture, we will review Extreme Value problem in single-variable calculus.

Extreme Values Part 1

The extreme value problem extends naturally to functions of multiple variables in multivariable calculus. Here's a list of related questions (for motivation only)

- **Mountain Climbing Problem**: If you're standing at a specific point on a hilly terrain, in which direction should you move to ascend as rapidly as possible?
- **Temperature Problem**: On a metal plate shaped like a given surface and heated in a certain way, where is the hottest or coolest point?
- **Multivariable Profit Maximization**: A company produces two different products, and the profit as a function of the number of each product produced is given. How many of each product should the company produce to maximize its profit?
- **Surface Area Problem**: What dimensions of an elliptical cylinder (an elongated can shape) with a given volume will minimize its surface area?
- **Optimal Control Problem**: For a system described by multiple parameters, which values of these parameters maximize or minimize a desired outcome (like efficiency)?

Before discussing the extreme value problem for multivariable calculus, let’s review this question for single variable calculus. We list the importan notions below. We will see the generalizations of them in the next lecture.

Let $f : \mathbb{R} \to \mathbb{R}$.

1. **Absolute maximum/minimum, Extreme values.**

   A function f has an *absolute maximum* at c if $f(c) \geqslant f(x)$ for all x in D, where D is the domain of f. The number $f(c)$ is called the *maximum value* of f on D. Similarly, f has an *absolute minimum* at c if $f(c) \leqslant f(x)$ for all x in D and the number $f(c)$ is called the *minimum value* of f on D. The maximum and minimum values of f are called the *extreme values* of f.

2. **Local maximum/minimum**

   A function f has a *local maximum* at c if $f(c) \geqslant f(x)$ when x is near c. Similarly, f has a *local minimum* at c if $f(c) \leqslant f(x)$ when x is near c.

3. **The Extreme Value Theorem**

   If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

4. **Fermat's Theorem**

   If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

5. **Critical Number**

   A *critical number* of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

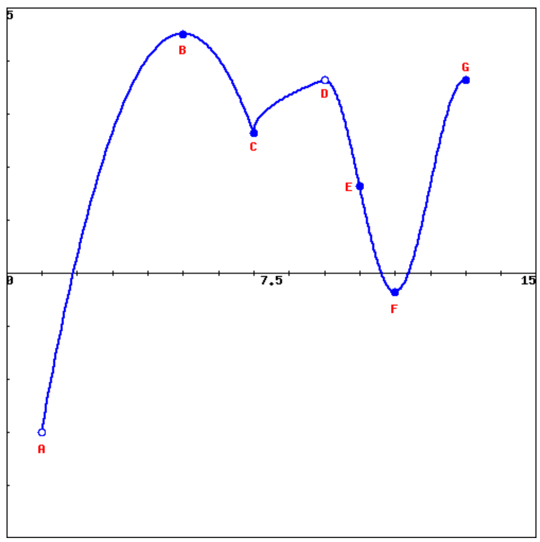
6. **The Closed Interval Method**

   To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

   1. Find the values of f at the critical numbers of f in (a, b).
   2. Find the values of f at the endpoints of the interval.
   3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 2

Identify the marked points as being an absolute maximum or minimum, a relative maximum or minimum, or none of the them.
local



Since A and D are not on the graph, they cannot be classified as absolute or relative extrema.

Point B is both a relative (local) and absolute maximum.

Points C and F are relative (local) minima (but not abosolute)

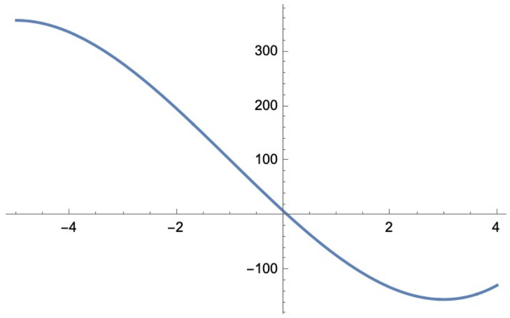
Point E is neither a maximum nor minimum.

Point G is a relative maximum but not absolute.

Example 3.

Consider the function $f(x) = 2x^3 + 6x^2 - 90x + 8, \quad -5 \leq x \leq 4.$

Find the absolute minimum and maximum value of this function.



ANS:

1. Find Critical Points.

Compute $f'(x) = 6x^2 + 12x - 90 = 0$

$\Rightarrow 6(x-3)(x+5) = 0 \Rightarrow x = 3$ and $x = -5$ are critical points

2. Evaluate f at the Endpoints and Critical points

$f(-5) = 358$

$f(4) = -128$

$f(3) = -154$

3. Determine Absolute Minimum and Maximum.

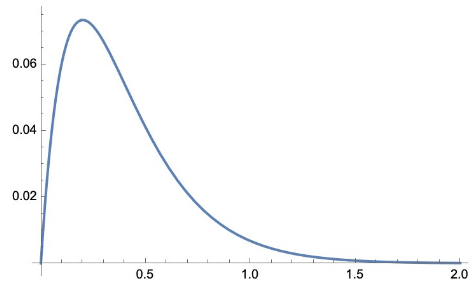
Compare the values above, we know the absolute minimum is $f(3) = -154$

Abs max is $f(-5) = 358$

Example 4.

Consider the function $f(x) = xe^{-5x}, \quad 0 \leq x \leq 2.$

Find the absolute minimum and maximum value of this function.



Solution.

Step 1. Find Critical Points.

Set $f'(x) = e^{-5x} - 5e^{-5x}x = 0$

$\Rightarrow \quad e^{-5x}(1-5x) = 0 \quad \Rightarrow x = \frac{1}{5}.$

Step 2. $f(0) = 0$

$f(2) = \frac{2}{e^{10}} > 0$

$f(\frac{1}{5}) = \frac{1}{5e} \approx 0.0735759.$

Step 3.

Abs max is $f(\frac{1}{5}) = \frac{1}{5e} \approx 0.0735759.$

Abs min is $f(0) = 0$

Example 5.

Find the absolute maximum and minimum values of $f(x) = \dfrac{x-4}{x+2}$, if any, over the interval $(-2, 4]$

Solution. Note

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{x-4}{x+2} = -\infty$$

Also

$$f'(x) = \frac{6}{(x+2)^2} > 0$$

Thus f has no critical points. and f is increasing as x increases.

So f will not have an absolute minimum

And f will have an absolute maximum at it's right end point $x=4$ by $f(4)=0$.

Exercise 6.

Find the extreme values of the function f on the interval $[0, \pi]$, and the x-value(s) at which they occur. If an extreme value does not exist, enter DNE for both the value and location.

$$f(x) = -10e^x \cos x$$

Solution.

Solution:
Set the derivative equal to zero to locate all critical numbers.

$$\begin{aligned} f'(x) = +10e^x \sin -10e^x \cos x &= 0 \\ -10e^x(-\sin x + \cos x) &= 0 \\ (-\sin x + \cos x) &= 0 \quad \text{since } e^x \neq 0 \text{ for any } x \\ \sin x &= \cos x \\ \tan x &= 1 \\ x &= \frac{\pi}{4} \quad \text{(the only solution in the given interval)} \end{aligned}$$

Find the value of f at each critical number and endpoint:

$$\begin{aligned} f(0) &= -10e^0 \cos(0) = -10 \\ f(\pi/4) &= -10e^{\pi/4} \cos(\pi/4) \approx -15.5088 \\ f(\pi) &= -10e^{\pi} \cos(\pi) = +10e^{\pi} \approx 231.407 \end{aligned}$$

The absolute minimum value is -15.5088 , located at $x = \pi/4$, and the absolute maximum value is 231.407 , located at $x = \pi$.

Exercise 7.

Determine where the absolute extrema of $f(x) = \dfrac{4x}{x^2+1}$ on the interval $[-4, 0]$ occur.

Solution.

1. **Find Critical Points:**

   Using the quotient rule, the derivative $f'(x)$ is:

$$\begin{aligned} f'(x) &= \frac{\left(x^2+1\right)(4) - (4x)(2x)}{\left(x^2+1\right)^2} \\ &= \frac{-4x^2+4}{\left(x^2+1\right)^2} \end{aligned}$$

   Notice that $\left(x^2+1\right)^2$ is never 0.

   Setting $f'(x) = 0$, we have $x = \pm 1$. Note that the specified inteval is $[-4, 0]$, so we only need to consider $x = -1$ as the critical point.

2. **Evaluate at the Endpoints and Critical Points:**

$$\begin{aligned} f(-1) &= \frac{4(-1)}{(-1)^2+1} = \frac{-4}{2} = -2 \\ f(-4) &= \frac{4(-4)}{(-4)^2+1} = \frac{-16}{17} \\ f(0) &= \frac{4(0)}{(0)^2+1} = 0 \end{aligned}$$

3. **Determine Absolute Minimum and Maximum:**

   Comparing these values, $f(-1) = -2$ is the smallest value, and $f(0) = 0$ is the largest value on the interval $[-4, 0]$.

   Therefore, the absolute minimum of $f(x)$ on the interval $[-4, 0]$ occurs at $x = -1$ with a value of $f(-1) = -2$, and the absolute maximum occurs at $x = 0$ with a value of $f(0) = 0$.